A Step-Wise Approach to Elicit Triangular Distributions

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Risk, Uncertainty & Estimating

"It is better to be approximately right rather than precisely wrong.

Warren Buffett

Outline

- Purpose of Presentation
- Background
  - The Uncertainty Spectrum
  - Expert Judgment Elicitation (EE)
  - Continuous Distributions
    - More details on Triangular, Beta & Beta-PERT Distributions
- Five Expert Elicitation (EE) Phases
- Example: Estimate Morning Commute Time
  - Expert Elicitation (EE) to create a Triangular Distribution
    - With emphasis on Phase 4’s Q&A with Expert (2 iterations)
  - Convert Triangular Distribution into a Beta-PERT
- Conclusion & Potential Improvements

Purpose of Presentation

Adapt / combine known methods to demonstrate an expert judgment elicitation process that …

1. **Models expert's inputs as a triangular distribution**
   - 12 questions to elicit required parameters for a bounded distribution
   - Not too complex to be impractical; not too simple to be too subjective
2. **Incorporates techniques to account for expert bias**
   - A repeatable Q&A process that is iterative & includes visual aids
   - Convert Triangular to Beta-PERT (if overconfidence was a
3. **Is s inp**

This paper will show one way of "extracting" expert opinion for estimating purposes. Nevertheless, as with most subjective methods, there are many ways to do this.

The Uncertainty Spectrum

No Estimate Required

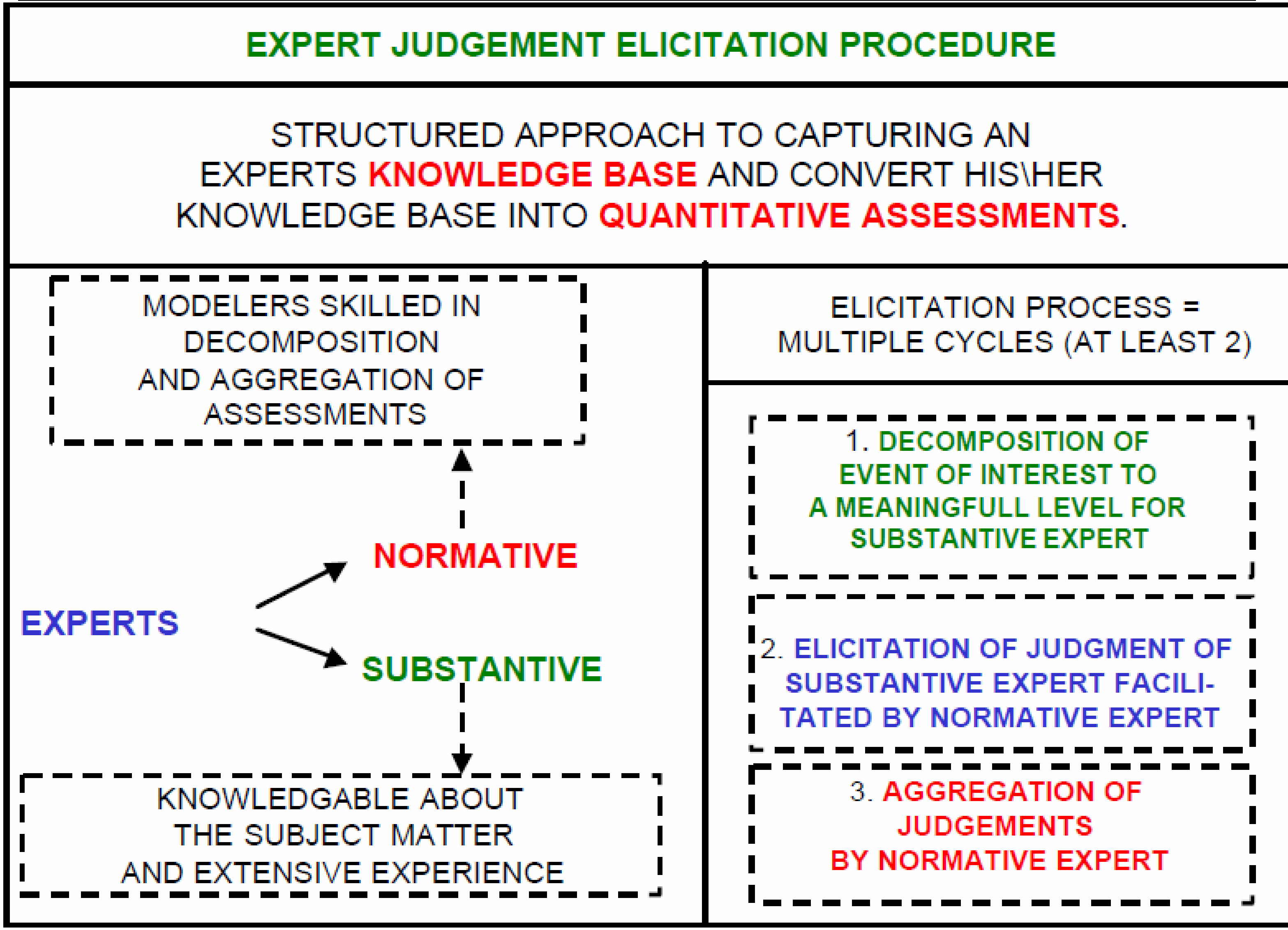
Total Certainty =	**Complete information**		**All known**
Specific Uncertainty		Objective Probabilities	**Data / Knowledge**
	Partial information		
Known unknowns			
General Uncertainty		Subjective Probabilities	**Expert Opinion**
Total Uncertainty =	**No information**		**Unknown unknowns**

No Estimate Possible

Reference: *Project Management Consulting* by AEW Services, 2001

Expert opinion is useful when little information is available for system requirements, system characteristics, durations & cost

Expert Judgment Elicitation (EE)



Source: Making Hard Decisions, An Introduction to Decision Analysis by R.T. Clemen

Triangular Distribution

- **Used in situations were there is little or no data**
  - Just requires the lowest (L), highest (H) and most likely values (M)

Each x-value has a respective f(x), sometimes called "Intensity" that forms the following PDF:

$$f(x) = \frac{2(x-L)}{(M-L)(H-L)}, \quad L \le x < M$$

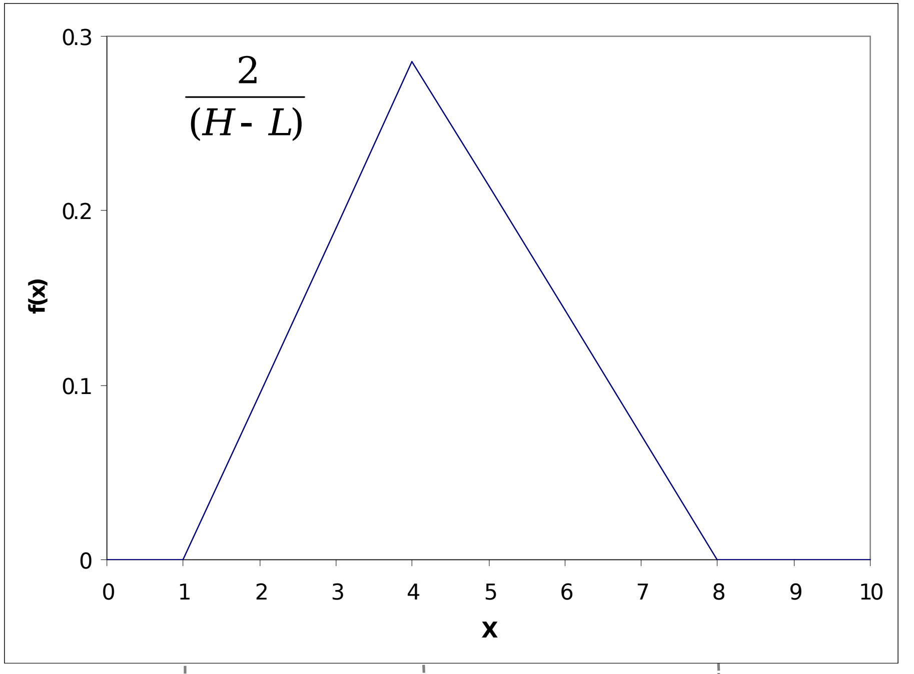
$$= \frac{2(H-x)}{(H-M)(H-L)}, \quad M \le x < H$$

$$= 0, \quad \text{otherwise}$$

L, M & H are all that's needed to calculate the Mean and Standard Deviation:

$$\mu = \frac{(L+M+H)}{3}$$

$$\sigma = \sqrt{\frac{(L^2 + M^2 + H^2 - L*M - L*H - M*H)}{18}}$$



L M
H

Beta Distribution

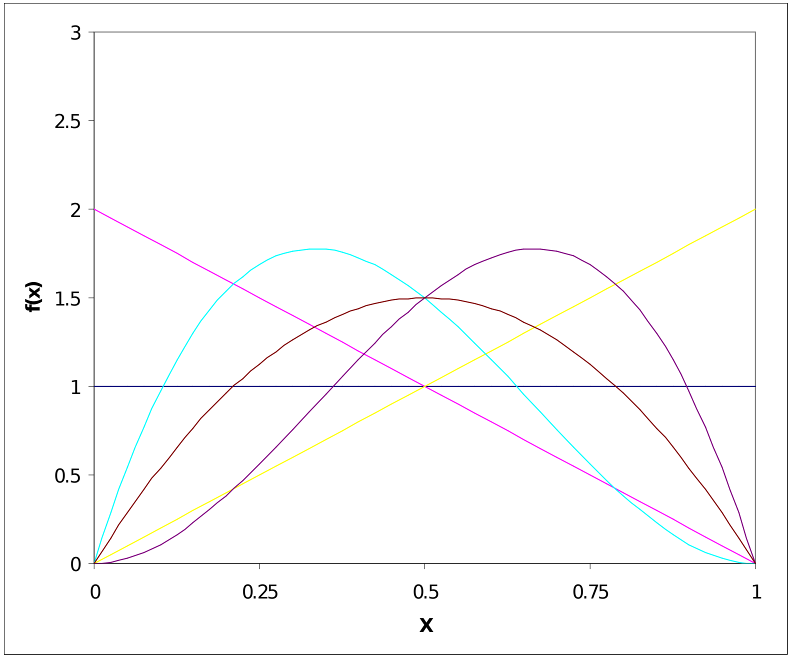
Bounded on [0,1] interval, scale to any interval & very flexible shape

$$f(x) = \left(\frac{1}{H-L}\right)\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\left(\frac{x-L}{H-L}\right)^{\alpha-1}\left(\frac{H-x}{H-L}\right)^{\beta-1} \quad L < x < H \qquad \text{Shape Parameters: } \alpha > 0, \beta > 0$$

$$= 0 \text{ otherwise}$$

Calculated Gamma values using Excel's GAMMALN function:

$$\left\{\begin{aligned} \Gamma(\alpha+\beta) &= \mathrm{EXP}[\mathrm{GAMMALN}(\alpha+\beta)] \\ \Gamma(\alpha) &= \mathrm{EXP}[\mathrm{GAMMALN}(\alpha)] \\ \Gamma(\beta) &= \mathrm{EXP}[\mathrm{GAMMALN}(\beta)] \end{aligned}\right.$$



Most schedule or cost estimates follow right skewed pattern. But how do we know a and b? Answer: Beta-PERT Distribution.

Sources: 1. Dr. Paul Garvey, *Probability Methods for Cost Uncertainty Analysis*, 2000

2. LaserLight Networks, Inc, "Beta Modeled PERT Schedules"

Beta-PERT Distribution

Requires lowest (L), highest (H) & most likely values (M)

Use L, M and H to calculate mean(m) and standard deviation (s) :

$$\mu = \frac{(L + \lambda * M + H)}{\lambda + 2} \qquad \sigma = \frac{(H - L)}{6}$$

Use L, H, m and s To calculate shape parameters, a & b :

$$\alpha = \frac{(\mu - L)}{(H - L)} * \left(\frac{(\mu - L)(H - \mu)}{\sigma^2} - 1 \right) \qquad \text{where } \alpha > 0, \beta > 0$$

$$\beta = \frac{(H - \mu)}{(\mu - L)} * \alpha$$

a and b are needed to define the Beta Function and compute the Beta Probability Density:

Beta Probability Density Function (as shown in slide 9):

$$f(x) = \left(\frac{1}{H - L} \right) \left(\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right) \left(\frac{x - L}{H - L} \right)^{\alpha - 1} \left(\frac{H - x}{H - L} \right)^{\beta - 1} \qquad L < x < H$$

Calculated Gamma values using Excel's GAMMALN function:

$$\Gamma(\alpha + \beta) = \text{EXP}[\text{GAMMALN}(\alpha + \beta)]$$
$$\Gamma(\alpha) = \text{EXP}[\text{GAMMALN}(\alpha)]$$
$$\Gamma(\beta) = \text{EXP}[\text{GAMMALN}(\beta)]$$

Sources: 1. Dr. Paul Garvey, *Probability Methods for Cost Uncertainty Analysis*, 2000

2. LaserLight Networks, Inc, "Beta Modeled PERT Schedules"

Expert Elicitation (EE) Phases

Expert Elicitation consists of five phases: *(note that Phases 4 & 5 are iterative)*

1. Motivating the expert
2. Training (conditioning) the expert
3. Structuring objective, assumptions & process
4. Assessing (encoding) expert's responses
   - Q&A – Expert's technical opinion is elicited

Our Example will emphasize the Phase 4 Q&A

5. Verifying encoded values &

Example: Estimate Commute Time

- **Why this example?**
  - Fairly easy to find a subject matter expert
  - It is a parameter that is measurable
  - Most experts can estimate a most likely time
  - Factors that drive uncertainty can be readily identified
  - People general care about their morning commute time!

Let's begin with Phase 1 … Motivating the Expert:

1. **Motivating the expert**
   - Explain the importance & reasons for collecting the data
   - Explore stake in decision & potential for motivational bias

EE Phase 2: *Commute Time*

2. Structuring objective, assumptions & process

- Be explicit about what you want to know & why you need to know it
  - *Clearly define variable & avoid ambiguity and explain data values that are required (e.g. hours, dollars, %, etc)*

The Interviewer should have worked with you to develop the Objective and up to 5 Major Assumptions in the table below

- Please resolve any questions or concerns about the Objective and/or Major Assumptions prior to continuing to "Instructions".

Objective: Develop uncertainty distribution associated with time (minutes) it will take for your morning commute starting 1 October 2014.

Assumption 1: Your commute estimate includes only MORNING driving time
Assumption 2: The commute will be analogous to the one you've been doing
Assumption 3 Period of commute will be from 1 Oct 2014 thru 30 Sep 2015

EE Phase 3: Commute Time

3. **Training (conditioning) the expert**
   - **Go over instructions for Q&A process**
   - **Emphasize benefits of time constraints & 2 iterations**

Instructions: This interview is intended to be conducted in two Iterations.
Each iteration should take no longer than 30 minutes.

A. Based on your experience, answer the 12 question sets below.
B. Once you've completed the questions, review them & take a 15 minute break.
C. Using the triangular graphic to assist you, answer all of the questions again.

Notes:

A. The 2nd iteration is intended to be a refinement of your 1st round answers.
B. Use lessons-learned from the 1st iteration to assist you in the 2nd iteration.
C. Your interviewer is here to assist you at any point in the interview process.

EE Phase 3: Commute Time (cont'd)

3. Training the expert (continued)

For 2 Questions, you'll need to provide your assessment of likelihood:

Descriptor	Explanation	Probability
Absolutely Impossible	No possibility of occurrence	Values will be defined by SME
Extremely Unlikely	Nearly impossible to occur; very rare	
Very Unlikely	Highly unlikely to occur; not common	
Indifferent between "Very Unlikely" & "Even chance"		
Even Chance	50/50 chance of being higher or lower	
Indifferent between "Very Likely" & "Even chance"		
Very Likely	Highly likely to occur; common occurrence	
Extremely Likely	Nearly certain to occur; near 100% confidence	
Absolutely Certain	100% Likelihood	

Example: Assume you estimated a "LOWEST" commute time of 20 minutes.

Your place a value = 10.0% as the probability associated with "Very Unlikely."

Therefore:

a) You believe it's "VERY UNLIKELY" your commute time will be less than 20 minutes, and

b) This is equal to a 10.0% chance that your commute time would be less than 20 min.

EE Phase 4: Commute Time (iteration 1)

Question 1: Expert creates "value-scale" tailored his/her bias …

What probability would you assign to a value that's "Very Unlikely"

What probability would you assign to a value that's "Extremely Unlikely"

Available Selection of Values to the Expert (shaded cells were selected by expert)**:**

VERY LIKELY	VERY UNLIKELY	EXTREMELY LIKELY	EXTREMELY UNLIKELY
80.0%	20.0%	96.0%	4.0%
82.5%	17.5%	97.0%	3.0%
85.0%	15.0%	98.0%	2.0%
87.5%	12.5%	98.5%	1.5%
90.0%	10.0%	99.0%	1.0%
92.5%	7.5%	99.5%	0.5%
95.0%	5.0%	99.9%	0.1%

EE Phase 4: Commute Time (iteration 1)

Question 1: Expert creates "value-scale" tailored his/her bias …

What probability would you assign to a value that's "Very Unlikely"

What probability would you assign to a value that's "Extremely Unlikely"

Descriptor	Explanation	Probability
Absolutely Impossible	No possibility of occurrence	0.0%
Extremely Unlikely	Nearly impossible to occur; very rare	1.0%
Very Unlikely	Highly unlikely to occur; not common	10.0%
Indifferent between "Very Unlikely" & "Even chance"		*30.0%*
Even Chance	50/50 chance of being higher or lower	50.0%
Indifferent between "Very Likely" & "Even chance"		*70.0%*
Very Likely	Highly likely to occur; common occurrence	90.0%
Extremely Likely	Nearly certain to occur; near 100% confidence	99.0%
Absolutely Certain	100% Likelihood	100.0%

Only 2 probabilities needed to be elicited in order to create a Value-Scale that has 9 categories!

EE Phase 4: Commute Time (Iteration 1)

4. Assessing expert's responses (Q&A)

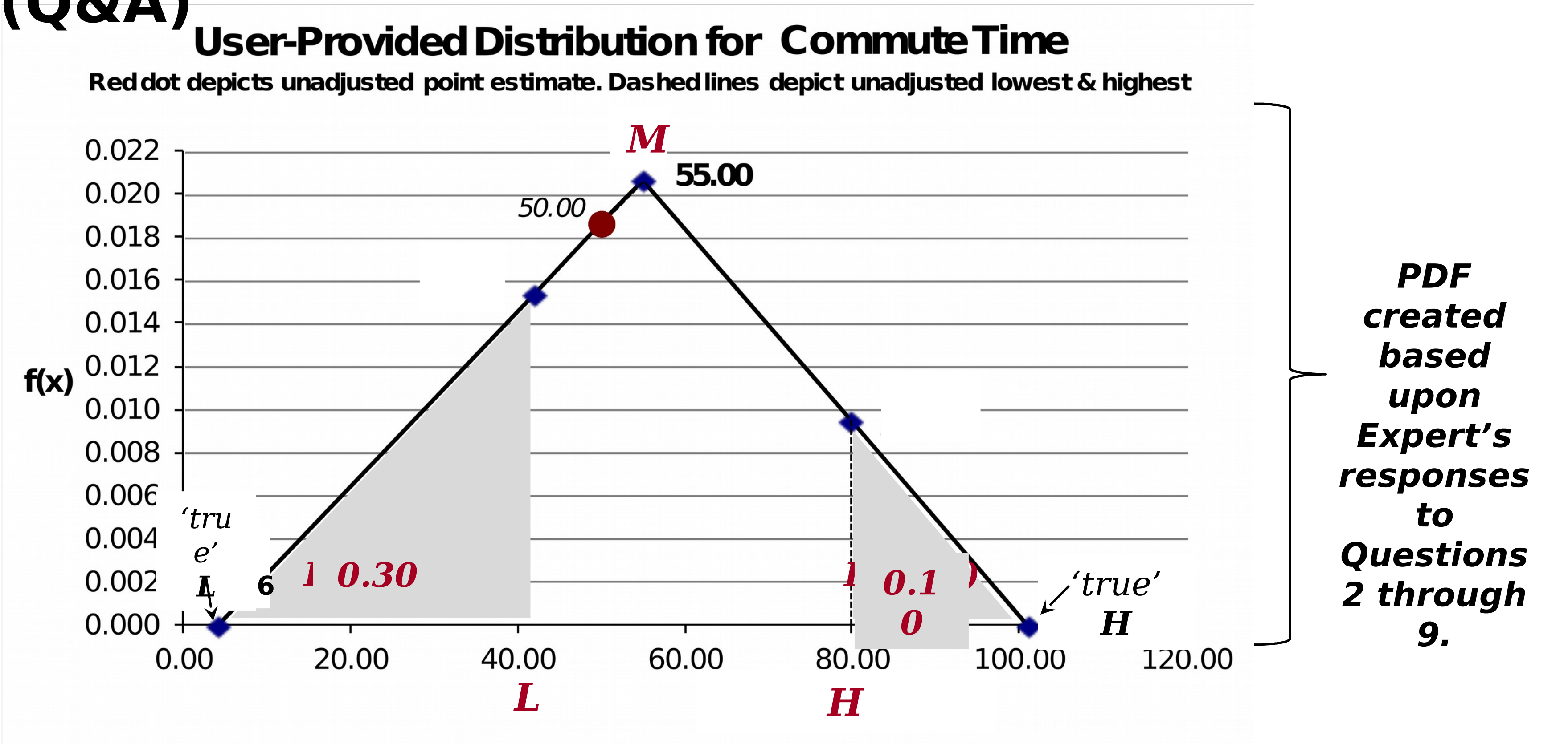
Given the objective and assumptions …

2. Describe input parameter (WBS 4): *Commute (in minutes) as Triangular Dist'n*
3. What's the Most Likely value, *M?* *50*
4. Adjust *M* (if applicable) *55*
5. What's the chance actual value could exceed *M?* *N/A*
6. What's the Lowest value, *L* *42*
7. What's the chance actual value could be less than *L?* *Indifferent-Low*
8. What's the Highest value, *H* *80*
9. W
   Ve

This 1st iteration tends to result in anchoring bias on *M*, over-confidence on *L* and *H*, and poor rationale

EE Phase 4: Commute Time (iteration 1)

4. Assessing expert's responses (Q&A)



<u>Given from Expert</u>: *L=42, M=55, H=80, p(x<L)=0.30* and ***p(x>H)=0.10***

<u>Calculation of 'true' L and H (a)</u>: *L = 1.56* and ***H = 101.15 … Do these #'s appear reasonable?***

(a) Method to solve for *L* and *H* presented in "*Beyond Beta*," Ch1 (The Triangular Distribution)

EE Phase 4: Commute Time (iteration 1)

Question 10: Expert & Interviewer brainstorm risk factors …

What risk factors contributed to the uncertainty in your estimate?

Weather
Accident(s)
Road Construction
Departure Time
Red Lights
Emergency vehicles
School buses
Not feeling well
Inexperienced driver
Unfamiliar with route

Create Risk Breakdown Structure (RBS)

Objective	Means	Barriers / Risks
Maximize Average Speed	Avoid Dense Traffic	Weather
		Accident(s)
		Road Construction
		Departure Time
	Avoid stops	Red Lights
		Emergency vehicles
		School buses
	Optimize driving	Not feeling well
		Inexperienced driver
		Unfamiliar with route

Question 11: Expert selects top 6 risk factors …

What are the top 6 risk factors that contributed to your estimate uncertainty?

User Input	Examples or Justification:
Weather	Rain, snow & especially ice, have caused major delays in the past; I expect similar impacts in 2014.
Accident(s)	Accidents occasionally occur. In some cases, these have added 60 minutes to my commute!
Road Construction	Sometimes road crews shut down 1 or 2 lanes; typically adding 10 - 20 minutes to my commute.
Departure Time	I try to leave 1 hour before rush hour. Leaving later can add 10-15 minutes to my commute.
Not Feeling Well	If I'm not feeling well, I'll drive more slowly or even make a wrong turn! Can add 5 min to commute.
Red Lights	I tend to "catch" the same lights every day so this factor could add 1-2 minutes to my commute.

EE Phase 4: Commute Time (iteration 1)

Question 12: Expert scores each risk factor's contribution to uncertainty …

Scor … g instruction:

If the specified risk factor: *		
is the largest contributor to uncertainty (e.g. biggest driver of H)	then score it a	**5.0**
Indifference		*4.5*
is a significant contributor to uncertainty (e.g. big driver of H)	then score it a	**4.0**
Indifference		*3.5*
has a moderate effect on uncertainty (e.g. nominal impact on H)	then score it a	**3.0**
Indifference		*2.5*
has a small effect on uncertainty (e.g. not a big driver of H)	then score it a	**2.0**
Indifference		*1.5*
is the smallest contributor to uncertainty (e.g. smallest driver of H)	then score it a	**1.0**

* Note: You can have 2 or more risk factors with a score of 5 (or score of 1).

Risk Factor	**Score**
Weather	**5.0**
Accident(s)	**5.0**
Road Construction	**2.0**
Departure Time	**4.0**
Not Feeling Well	**1.0**
Red Lights	**1.5**

Expert provides a score for each risk factor (rationale not shown).

The 1st iteration of Q&A is complete. Recommend the expert take a 15 minute break before re-starting Q&A

EE Phase 4: Commute Time (Iteration 2)

4. Assessing expert's responses (Q&A)

Given the objective, assumptions & input parameter (WBS4):

1. Do you need to modify the probability value scale?
   No
2. Do you need to re-characterize the input parameter?
   No
3. Do you want to adjust your Most Likely Value, *M*? *No*
4. What's the chance the actual value could exceed *M?*
   N/A

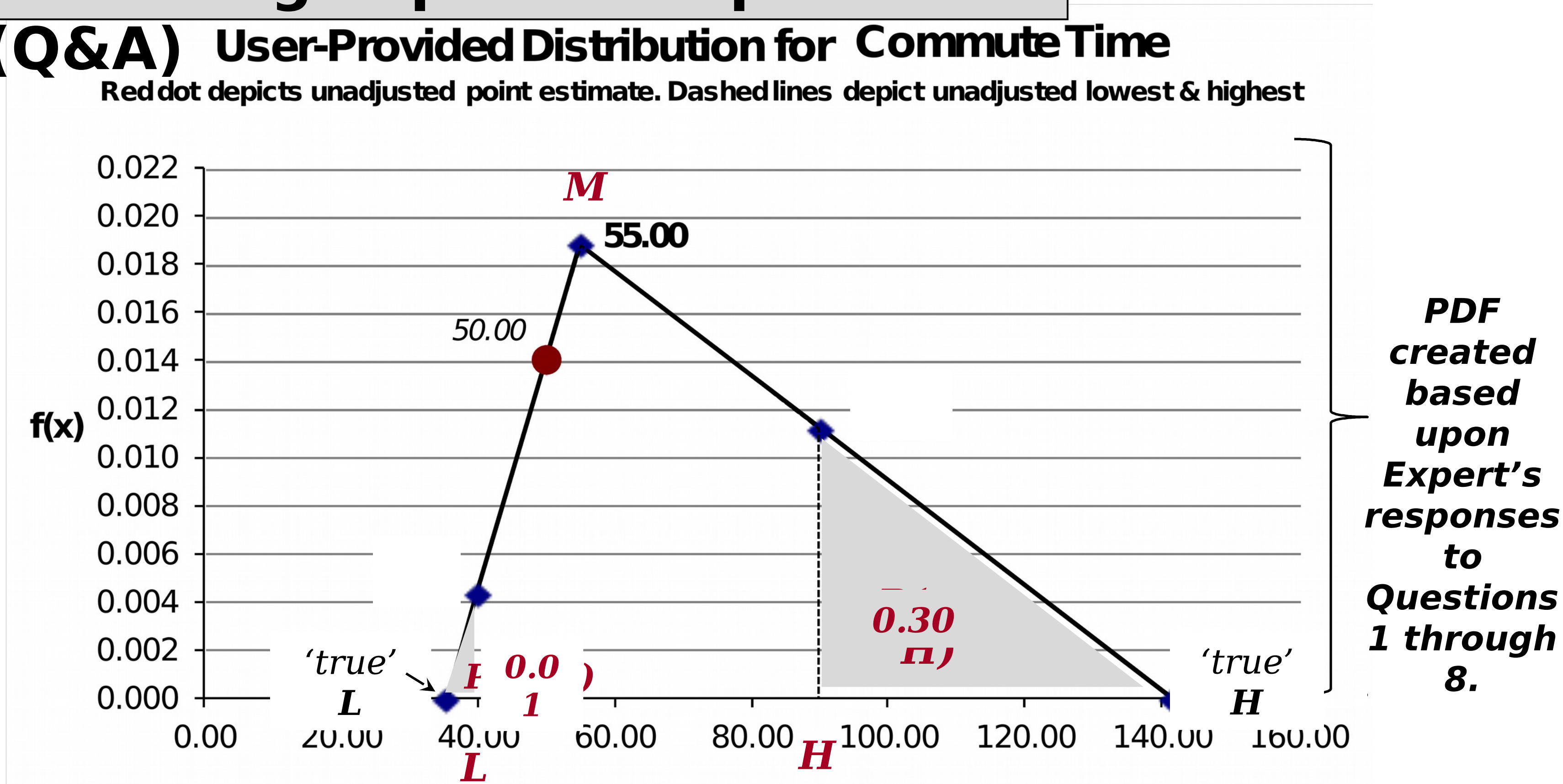
Assuming best case: weather, accidents, road const, departure time, etc.
Document Scenario

5. What's the Lowest value, *L* *40*
6. What's the chance actual value could be less than *L?*
   Extremely Unlikely

Assuming worst case: weather, accidents, road const, departure time, etc.

EE Phase 4: Commute Time (iteration 2)

4. Assessing expert's responses (Q&A)



Given from Expert: ***L=40, M=55, H=90, p(x<L)=0.10*** and ***p(x>H)=0.30***

Calculation of 'true' L and H [a]: ***L = 35.44*** and ***H = 143.92*** … ***Do these #'s appear***

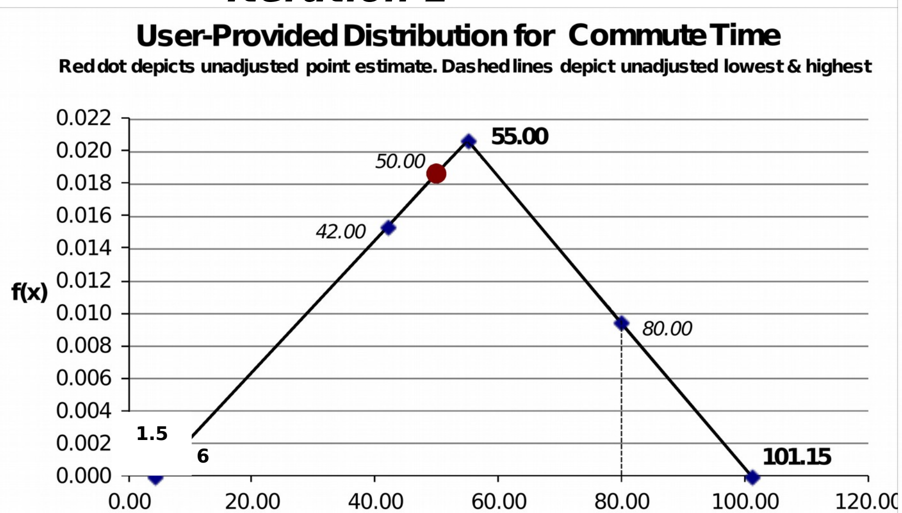
2nd iteration helps "condition" expert to reduce anchoring bias on *M*, counter over-confidence on *L* & *H*, calibrate 'values' & improve rationale.

(a) Method to solve for L and H presented in "Beyond Beta," Ch1 (The Triangular Distribution)

EE Phase 5: Commute Time (iteration 2)

5. Verifying encoded values & documentation

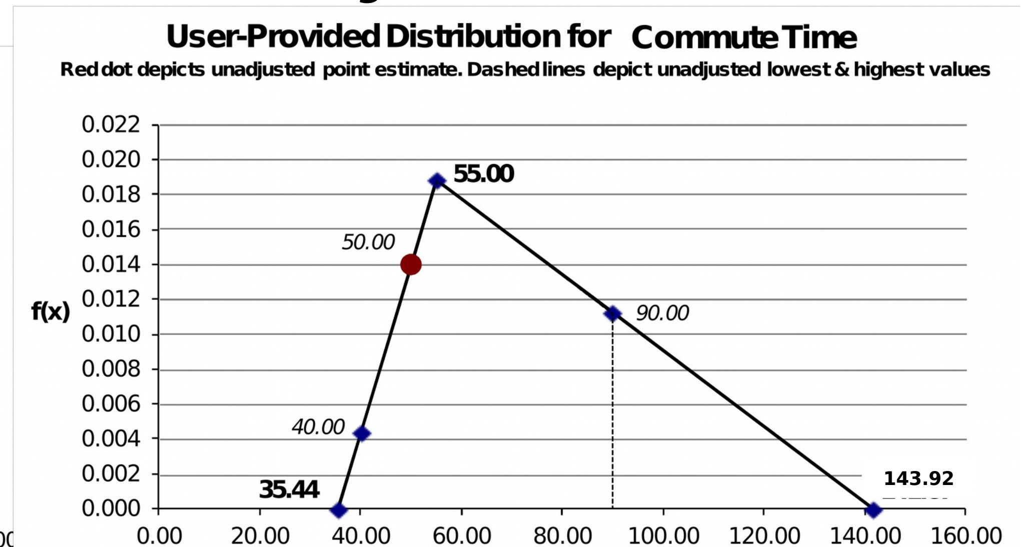
Triangular PDF from Iteration 1



L =1.56 H = 101.15

Inputs not necessarily sensitive to risk factors => Optimistic Bias

Triangular PDF from

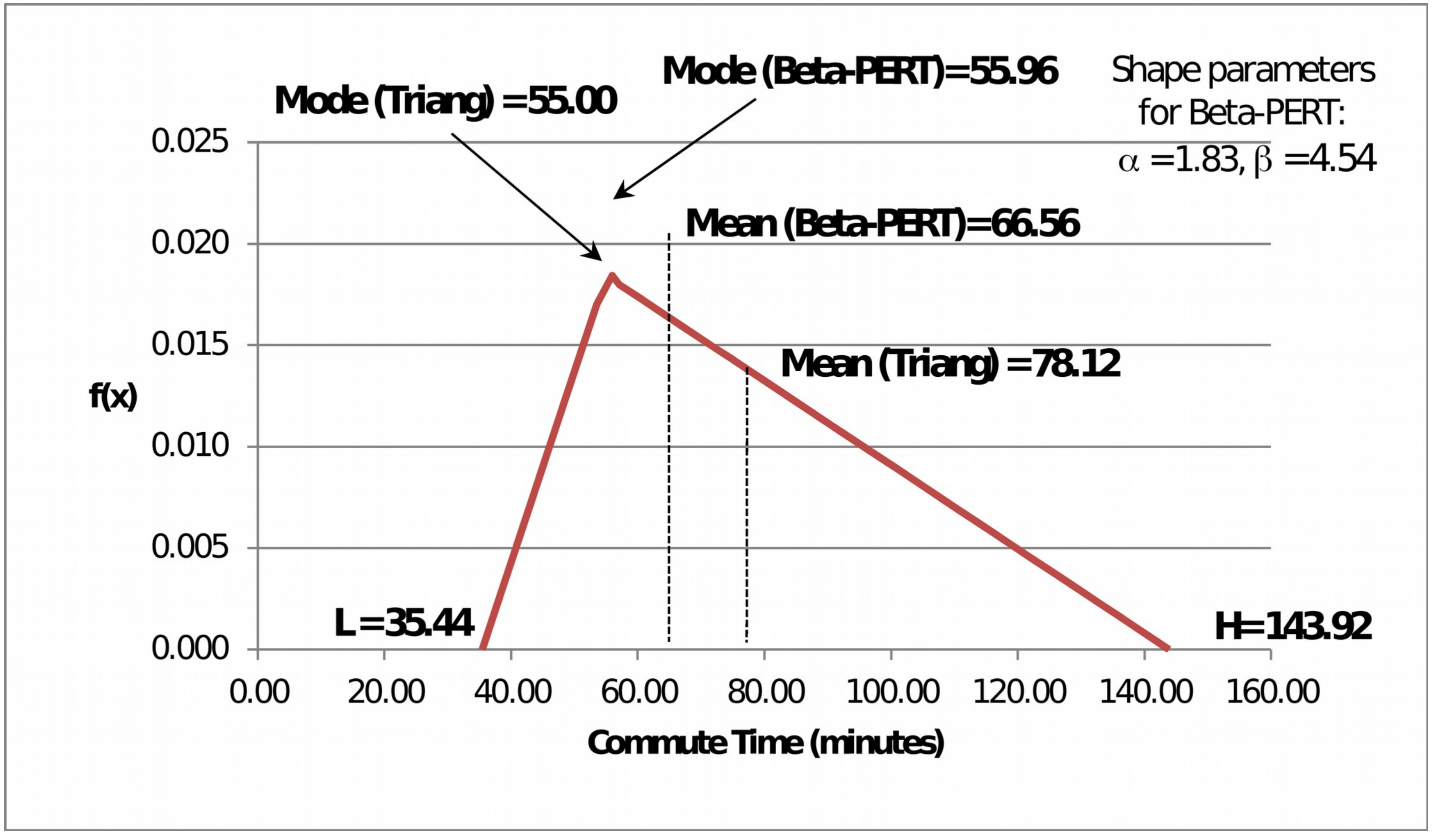


L =35.44 H = 143.92

Inputs sensitive to weighted risk factors => Minimum-Bias

The 2nd iteration helped elicit an *L* that seems feasible and an *H* that accounts for worst-case risk factors

Results (Triangular & Beta-PERT)



- **In most cases, Beta-PERT is preferred (vs triangular)**
  - Beta-PERT's mean is only slightly greater than its mode
- **However, triangular would be preferred (vs Beta-PERT) if elicited data seems to depict over-confidence (e.g. H value is optimistic)**
  - Triangular PDF compensates for this by 'exaggerating' the mean value

Conclusion

We provided an expert elicitation overview that …

1. **Demonstrated a way to model expert opinion as a triangular distribution**
   - A process that **does not "over-burden"** the subject matter expert
2. **Incorporated techniques to address expert bias**
   - Iterative Q&A process that includes use of visual aids
   - Relied on at least a **2nd iteration** to help minimize inaccuracy & bias
   - Convert Triangular to Beta-PERT (if overconfidence was addressed)
3. **Structured the process to help justify expert's inputs**
   - Rationale required for each response
   - RBS to help identify what **risk factors** contribute to uncertainty

Potential Improvements

- More upfront work on “training” the expert
- Criteria when to elicit mean or median (vs mode)
- Add 2 questions to create Modified Beta-PERT
- Improve scaling tables that depict expert’s judgment
- Create “starter” Risk Breakdown Structures”
  - Facilitates brainstorming process of possible risk factors
- Convert best case & worst case scenarios to probabilities
- Improve method of weighting risk factors
-

So … hopefully … this adds to the conversation on how best to leverage expert judgment in the cost community.

Intuition versus Analysis

Quickly answer the question:

"A bat and a ball cost $ 1.10 in total.
The bat costs $1 more than the ball.
How much does the ball cost?."

Sources not Referenced in Presentation

1. Liu, Y., "Subjective Probability," Wright State University.

2. Kirkebøen, G., "Decision behaviour – Improving expert judgement, 2010

3. Vose, D., *Risk Analysis (2nd Edition)*, John Wiley and Sons, 2004

4. "Expert Elicitation Task Force White Paper," US EPA, 2009

5. Clemen, R.T. and Winkler, R.L. (1990) Unanimity and compromise among probability forecasters. Management Science 36 767-779

A Step-Wise Approach to Elicit Triangular Distributions

Formerly entitled "An Elicitation Method to Generate Minimum-Bias Probability Distributions"

Questions?

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Probability Distributions

Bounded

- **Triangular & Uniform**
- **Histogram**
- **Discrete & Cumulative**
- **Beta & Beta-PERT**

Unbounded

- **Normal & Student-t**
- **Logistic**

Left bounded

- **Lognormal**
- **Weibull & Gamma**
- **Exponential**
- **Chi-square**

Non-Parametric Distributions: Mathematics defined by the shape that is required. Empirical, intuitive and easy to understand.

Parametric Distributions: Shape is born of the mathematics describing theoretical problem. Model-based. Not usually intuitive.

Of the many probability distributions out there, Triangular & Beta-PERT are among the most popular used for expert elicitation

Reasons For & Against Conducting EE

Reasons for Conducting an Expert Elicitation

- The problem is complex and more technical than political
- Adequate data (of suitable quality and relevance) are unavailable or unobtainable in the decision time framework
- Reliable evidence or legitimate models are in conflict
- Qualified experts are available & EE can be completed within decision timeframe
- Finances and expertise are sufficient to conduct a robust & defensible EE

Reasons Against Conducting and Expert Elicitation

- The problem is more political than technical
- A large body of empirical data exists with a high degree of consensus
- Findings of an EE will not be considered legitimate or acceptable by stakeholders
- Information that EE could provide is not critical to the assessment or decision
- Cost of obtaining EE info is not commensurate with its value in decision-making
- Finances and/or expertise are insufficient to conduct a robust & defensible EE
- Other acceptable methods or approaches are available for obtaining the needed information that are less intensive and expensive

Some Common Cognitive Biases

- Availability
  - Base judgments on outcomes that are more easily remembered
- Representativeness
  - Base judgments on similar yet limited data and experience. Not fully considering other relevant, accessible and/or newer evidence
- Anchoring and adjustment
  - Fixate on particular value in a range and making insufficient adjustments away from it in constructing an uncertainty estimate
- Overconfidence (sometimes referred to as Optimistic bias)
  - Strong tendency to be more certain about one's judgments and conclusions than one has reason. Tends to produce optimistic bias.
- Control (or "Illusion of Control")